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Gravitational Energy from a Quadratic Lagrangian with Torsion[1]

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Abstract

Gravitation is considered as a gauge field within the formalism of Utiyama and Kibble. In empty space-time a Lagrangian density, quadratic in Riemann's curvature tensor and in Cartan's torsion tensor, is introduced. The equations of motion are coupled differential equations for the curvature and torsion tensors. The spin of the torsion field behaves as a curvature source and the energy of both fields acts as a torsion source. Each field has an energy tensor, similar to the Maxwell tensor of electrodynamics, vanishing in a torsionless space. It thus appears that the torsion of space-time is a geometric property that makes possible the propagation of gravitational energy in the absence of matter.

1. *Introduction*

Electrodynamics has always been considered as the ideal model of a field theory. It describes the energy flow in empty space by means of Maxwell's tensor, which is a quadratic expression in the electric and magnetic fields. As is well known, this tensor is canonically obtained from the free electromagnetic Lagrangian density

$$\mathscr{L}_{\text{em}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.1}$$

which is also quadratic in the fields.

Einstein's gravitational theory may, in turn, be derived from Palatini's Lagrangian

$$\mathscr{L}_g = (-g)^{1/2}R \tag{1.2}$$

The analog of $F^{\mu\nu}$ in this approach is Christoffel's connection $\Gamma^{\lambda}{}_{\mu\nu}$. In

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fact, Lagrangian (1.2) is equivalent, up to a four-divergence, to the following quadratic expression in $\Gamma^{\lambda}{}_{\mu\nu}$:

$$\mathscr{L}^*{}_g = (-g)^{1/2} g^{\mu\nu}(\Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha} - \Gamma^{\alpha}{}_{\mu\nu}\Gamma^{\beta}{}_{\alpha\beta}) \tag{1.3}$$

The standard procedure gives, from this Lagrangian, the following pseudotensor (Weber, 1971, p. 73):

$$t_{\mu}{}^{\nu} = (-g)^{-1/2} [\mathscr{L}^*{}_g \delta_{\mu}{}^{\nu} - g^{\rho\beta}(\partial \mathscr{L}^*{}_g / \partial g^{\rho\beta}{}_{,\nu})] \tag{1.4}$$

which represents the potential energy of a test mass in a gravitational field (Eddington, 1924, p. 135). It is associated to the forces that cause the ordinary (noncovariant) acceleration of the particle. This interpretation amounts to establish a parallelism between the Lorentz equation governing the motion of a charged particle immersed in an electromagnetic field, and the geodesic equation in general relativity. Although this analogy discloses the similarity of the motion of charges and masses, it fails to unveil the dynamic structure of the gravitational field. The very meaning of the geodesic concept is to conceive the motion of a test mass as purely inertial, i.e., with a zero covariant acceleration. The gravitational field exhibits its dynamical content by means of the tidal forces, which are the origin of the relative acceleration of two free neighboring masses. This is clearly expressed by the equation of geodesic deviation

$$D^2\eta^{\alpha}/D\tau^2 + R^{\alpha}{}_{\beta\gamma\delta}(dx^{\beta}/d\tau)\eta^{\gamma}(dx^{\delta}/d\tau) = 0 \tag{1.5}$$

which is the proper physical analog of the Lorentz equation of electromagnetism. Therefore the gravitational field is better characterized by Riemann's curvature tensor than by Christoffel's connexion.

This correspondence between the electromagnetic field tensor $F^{\mu\nu}$ and the curvature tensor $R^{\mu}{}_{\nu\rho\sigma}$ appears in a natural way in the context of gauge field theories. As was shown by Weyl (1931, p. 89), the electromagnetic field compensates the noninvariance of a charged particle Lagrangian under local (coordinate dependent) phase transformations. In the same fashion, the gravitational field emerges as the compensating field associated to local Lorentz transformations (Weyl, 1929; Utiyama, 1956).

When this invariance is extended to the full 10-parameter group of inhomogeneous Lorentz transformations (Poincaré group), in addition to Riemann's tensor, the gauge field is also constituted by Cartan's torsion tensor $C^{\mu}{}_{\nu\rho}$ (Kibble, 1961).

According to the general theory of gauge fields, the free gravitational Lagrangian must be a function only of the potentials and the tensor fields. By analogy with electrodynamics it should be a quadratic function in the curvature and torsion. Lagrangians quadratic in Riemann's tensor have been introduced by several authors (Pauli, 1919; Lanczos, 1938; Carmelli, 1972). They essentially reproduce Einstein's equation in empty space-time. However, as we show in Section 5, the proper analog of Maxwell's energy tensor in this approach, equation (5.4), vanishes in the absence of matter. Hayashi (1968) considered a Lagrangian quadratic in Riemann's and Cartan's tensors, but he imposed certain subsidiary conditions that allowed him to recover Einstein's equation of gravitation.

In this paper we adopt a free gravitational Lagrangian, quadratic both in Riemann's and Cartan's tensors. It leads to first-order coupled differential equations for $R^{\mu}{}_{\nu\rho\sigma}$ and $C^{\mu}{}_{\nu\rho}$, each field acting as a source of the other. This relation resembles the coupling between the electric and magnetic fields in an electromagnetic wave. The curvature-field energy is still given by equation (5.4) and a similar expression accounts for the torsion-field energy $T_C{}^{\mu\nu}$ [equation (5.5)]. However, in the presence of torsion, the tensor $T_R{}^{\mu\nu}$ is no longer zero and the same is true for the homologous term $T_C{}^{\mu\nu}$.

We conclude that only spaces with torsion possess dynamical gravitational energy and that this energy is shared by the torsion and curvature fields.

The plan of this paper is the following: In Section 2 we shortly review the basic ideas of Kibble's paper and introduce the notation. In Section 3 we identify, with the help of Noether's theorem, the energy and spin of the sources, including the contribution of the gravitational field. In Section 4 we introduce the free gravitational Lagrangian and derive the corresponding equations of motion. Finally, in Section 5, we rewrite these equations in a full covariant language and show that the tensors $T_R{}^{\mu\nu}$ and $T_C{}^{\mu\nu}$ correspond to the gravitational energy of the dynamical fields.

2. *Gauge Approach to Gravitation*

Let us consider, following Kibble (1961), a Lagrangian density $\mathscr{L}(\chi, \partial_\mu \chi)$ representing all fields besides gravitation. Special relativity demands its invariance under the ten-parameter Poincaré transformations

$$\delta x^\mu = \epsilon^\mu{}_\nu x^\nu + \epsilon^\mu \tag{2.1}$$

$$\delta \chi(x) = \tfrac{1}{2}\epsilon^{\mu\nu} S_{\mu\nu} \chi(x) \tag{2.2}$$

where $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$. The matrices $S_{\mu\nu}$ satisfy

$$S_{\mu\nu} = -S_{\nu\mu} \tag{2.3}$$

$$[S_{\mu\nu}, S_{\rho\sigma}] = \eta_{\nu\rho} S_{\mu\sigma} + \eta_{\mu\sigma} S_{\nu\rho} - \eta_{\nu\sigma} S_{\mu\rho} - \eta_{\mu\rho} S_{\nu\sigma} \tag{2.4}$$

with $\eta_{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$. These commutation relations characterize the Lie algebra of the Lorentz group. According to the locality condition of Yang and Mills (1954), the physical system must also be invariant when the ten parameters $\epsilon^{\mu\nu}$ and ϵ^μ are replaced by ten arbitrary functions of the space-time coordinates, i.e.,

$$\delta x^\mu = \epsilon^\mu{}_\nu(x) x^\nu + \epsilon^\mu(x)$$

$$\delta \chi(x) = \tfrac{1}{2}\epsilon^{ij}(x) S_{ij} \chi(x) \tag{2.5}$$

Here both Latin and Greek indices run from 0 to 3. Two kinds of indices are needed to account for the fact that now the pure internal transformations are a subgroup of the extended Poincaré group, namely,

$$\epsilon^{\mu}{}_{\nu}(x) \not\equiv 0, \qquad \delta x^{\mu} \equiv 0, \qquad \delta\chi(x) \not\equiv 0$$

It should be noted that the requirement of invariance under the generalized Poincaré transformations corresponds to the equivalance principle of Einstein's gravitational theory (local validity of special relativity).

The original Lagrangian density $\mathscr{L}(\chi, \partial_\mu\chi)$ is no longer invariant under the extended Poincaré transformations (2.5), because the derivative $\chi_{,\mu}$ transforms as

$$\delta\chi_{,\mu} = \tfrac{1}{2}\epsilon^{ij}S_{ij}\chi_{,\mu} + \tfrac{1}{2}\epsilon^{ij}{}_{,\mu}\, S_{ij}\chi - \xi^{\nu}{}_{,\mu}\chi_{,\nu} \tag{2.6}$$

The invariance is recovered when we perform the customary "minimal" replacement of the ordinary derivative by a covariant one:

$$\chi_{;k} \equiv h_k{}^{\mu}(\chi_{,\mu} + \tfrac{1}{2}A^{ij}{}_{\mu}S_{ij}\chi) \tag{2.7}$$

In this expression ten vector gauge fields have been introduced to compensate the non invariance of the kinetic energy terms. They are the six antisymmetric $A^{ij}(x)$, associated to the internal Lorentz transformations $\epsilon^{ij}(x)$, and the four $h_k{}^{\mu}(x)$, originated on the displacements $\epsilon^{\mu}(x)$. The transformation properties of these fields are

$$\delta A^{ij}{}_{\mu} = \epsilon^{i}{}_{k}A^{kj}{}_{\mu} + \epsilon^{j}{}_{k}A^{ik}{}_{\mu} - \xi^{\nu}{}_{,\mu}A^{ij}{}_{\nu} - \epsilon^{ij}{}_{,\mu} \tag{2.8}$$

$$\delta h_k{}^{\mu} = \xi^{\mu}{}_{,\nu}h_k{}^{\nu} - \epsilon^{i}{}_{k}h_i{}^{\mu} \tag{2.9}$$

whence the covariant derivative transforms according to

$$\delta\chi_{;k} = \tfrac{1}{2}\epsilon^{ij}S_{ij}\chi_{;k} - \epsilon^{i}{}_{k}\chi_{;i} \tag{2.10}$$

from which the invariance of the Lagrangian density is guaranteed.

The commutator of the mixed second covariant derivatives yields

$$\chi_{;kl} - \chi_{;lk} = \tfrac{1}{2}R^{ij}{}_{kl}S_{ij}\chi - C^{i}{}_{kl}\chi_{;i} \tag{2.11}$$

where

$$R^{i}{}_{jkl} = h_k{}^{\mu}h_l{}^{\nu}(A^{i}{}_{j\mu,\nu} - A^{i}{}_{j\nu,\mu} - A^{i}{}_{k\mu}A^{k}{}_{j\nu} + A^{i}{}_{k\nu}A^{k}{}_{j\mu}) \tag{2.12}$$

$$C^{i}{}_{kl} = (h_k{}^{\mu}h_l{}^{\nu} - h_l{}^{\mu}h_k{}^{\nu})(b^{i}{}_{\mu,\nu} + A^{i}{}_{r\nu}b^{r}{}_{\mu}) \tag{2.13}$$

In equation (2.13) we have introduced the four quantities $b^{i}{}_{\mu}$ defined by

$$\begin{aligned} b^{i}{}_{\mu}h_i{}^{\nu} &= \delta_{\mu}{}^{\nu} \\ b^{i}{}_{\mu}h_j{}^{\nu} &= \delta^{i}{}_{j} \end{aligned} \tag{2.14}$$

that is to say, $b^{i}{}_{\mu}$ is the inverse of $h_i{}^{\mu}$.

To understand the physical meaning of these formulas let us consider the well-known similar relations describing the minimal electromagnetic interaction of a charged scalar field $\phi(x)$:

$$\phi_{;\mu} = \phi_{,\mu} - ieA_\mu \tag{2.15}$$

$$\phi_{;\mu\nu} - \phi_{;\nu\mu} = eF_{\mu\nu} \tag{2.16}$$

Comparing equations (2.15) and (2.16) with the homologous equations (2.7) and (2.11), we convince ourselves that the gravitational field is characterized by two local tensors $R^{ij}{}_{kl}$ and $C^i{}_{kl}$. The geometrical content of these tensors is made clear when we interpret the four fields $h_k{}^\mu(x)$ as the contravariant components of a vierbein system in a curved space. Correspondingly, the inverses $b^i{}_\mu(x)$ are the covariant components of the same tetrad. This identification provides the space-time with a metric given by the symmetric fundamental tensor

$$g_{\mu\nu} = b^k{}_\mu b_{k\nu} \tag{2.17}$$

Thus, the Greek indices, which refer to global components, are raised and lowered with the help of $g_{\mu\nu}$, whereas the Latin ones are handled with the local Minkowski tensor η_{ik}.

In addition to the metric, an affine connexion $\Gamma^\lambda{}_{\mu\nu}$, consistent with the structure of the covariant derivative (2.7), may also be introduced, namely,

$$\Gamma^\lambda{}_{\mu\nu} = h_i{}^\lambda(b^i{}_{\mu,\nu} + A^i{}_{k\nu}b^k{}_\mu) \tag{2.18}$$

Although the metric tensor (2.17) has a vanishing covariant derivative, expression (2.18) is not a Christoffel connection, because it is not symmetric. According to this geometric interpretation, the quantities $R^{ij}{}_{kl}$ and $C^i{}_{kl}$ are identified as the local components of the curvature and torsion tensors. The corresponding world tensors are

$$R^\rho{}_{\sigma\mu\nu} = \Gamma^\rho{}_{\sigma\mu,\nu} - \Gamma^\rho{}_{\sigma\nu,\mu} - \Gamma^\rho{}_{\lambda\mu}\Gamma^\lambda{}_{\sigma\nu} + \Gamma^\rho{}_{\lambda\nu}\Gamma^\lambda{}_{\sigma\mu} \tag{2.19}$$

$$C^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \tag{2.20}$$

Therefore, when we impose invariance of a given field with respect to independent Poincaré transformations at each space-time point, two gauge fields are needed. They have the geometric meaning of curvature and torsion of a non-Riemannian space. Together they represent the gravitational field.

3. *Sources of the Gravitational Field*

In this section we make a general analysis of the field equations and their sources, in order to identify the conserved currents that give rise to the curvature and torsion fields. We start from an arbitrary Lagrangian density

$$\mathscr{L}\{\chi, \partial_\mu\chi, h_k{}^\mu, \partial_\nu h_k{}^\mu, A^{ij}{}_\mu, \partial_\nu A^{ij}{}_\mu\} \tag{3.1}$$

assumed invariant under the extended Poincaré transformations (2.5). Direct application of Noether's theorem gives

$$\partial_\mu \{[\mathscr{L}\delta^\mu{}_\nu - (\partial\mathscr{L}/\partial\chi_{,\mu})\chi_{,\nu} - (\partial\mathscr{L}/\partial h_k{}^\rho{}_{,\mu})h_k{}^\rho{}_{,\nu} - (\partial\mathscr{L}/\partial A^{ij}{}_{\rho,\mu})A^{ij}{}_{\rho,\nu}]\,\delta x^\nu + (\partial\mathscr{L}/\partial\chi_{,\mu})\delta\chi + (\partial\mathscr{L}/\partial h_k{}^\rho{}_{,\mu})\delta h_k{}^\rho + (\partial\mathscr{L}/\partial A^{ij}{}_{\rho,\mu})\delta A^{ij}{}_\rho\} = 0 \tag{3.2}$$

where, for short, we have replaced inside the brackets the partial derivative symbol by a comma. Inserting in equation (3.2) the expressions for the variations given by equations (2.5), (2.8), and (2.9) and taking into account that ξ^ν, $\partial_\rho\xi^\nu$, $\partial_\alpha\partial_\beta\xi^\nu$, ϵ^{ij}, $\partial_\mu\epsilon^{ij}$, and $\partial_\alpha\partial_\beta\epsilon^{ij}$ should be considered as independent arbitrary functions antisymmetric in ij and symmetric in $\alpha\beta$, we obtain

$$\partial_\rho F_{ij}{}^{\mu\rho} = S^\nu{}_{ij} \tag{3.3}$$

$$F_{ij}{}^{\alpha\beta} = -F_{ij}{}^{\beta\alpha} \tag{3.4}$$

$$\partial_\mu S^\mu{}_{ij} = 0 \tag{3.5}$$

$$\partial_\mu G^{\mu\rho}{}_\nu = -T^\rho{}_\nu + \partial_\mu(F_{ij}{}^{\rho\mu}A^{ij}{}_\nu) \tag{3.6}$$

$$G^{\alpha\beta}{}_\nu + G^{\beta\alpha}{}_\nu = (F_{ij}{}^{\alpha\beta} - F_{ij}{}^{\beta\alpha})A^{ij}{}_\nu \tag{3.7}$$

$$\partial_\mu T^\mu{}_\nu = 0 \tag{3.8}$$

where we have defined

$$F_{ij}{}^{\mu\rho} \equiv \partial\mathscr{L}/\partial A^{ij}{}_{\mu,\rho} \tag{3.9}$$

$$S^\mu{}_{ij} \equiv (\partial\mathscr{L}/\partial\chi_{,\mu})\tfrac{1}{2}S_{ij}\chi - \tfrac{1}{2}[(\partial\mathscr{L}/\partial h^{j\rho}{}_{,\mu})h_i{}^\rho - (\partial\mathscr{L}/\partial h^{i\rho}{}_{,\mu})h_j{}^\rho] + \tfrac{1}{2}[(\partial\mathscr{L}/\partial A^{ik}{}_{\rho,\mu})A_j{}^k{}_\rho - (\partial\mathscr{L}/\partial A^{jk}{}_{\rho,\mu})A_i{}^k{}_\rho] + \tfrac{1}{2}[(\partial\mathscr{L}/\partial A^{ki}{}_{\rho,\mu})A^k{}_{j\rho} - (\partial\mathscr{L}/\partial A^{kj}{}_{\rho,\mu})A^k{}_{i\rho}] \tag{3.10}$$

$$G^{\alpha\beta}{}_\nu \equiv (\partial\mathscr{L}/\partial h_k{}^\nu{}_{,\alpha})h_k{}^\beta \tag{3.11}$$

$$T^\mu{}_\nu \equiv \mathscr{L}\delta^\mu{}_\nu - (\partial\mathscr{L}/\partial\chi_{,\mu})\chi_{,\nu} - (\partial\mathscr{L}/\partial h_k{}^\rho{}_{,\mu})h_k{}^\rho{}_{,\nu} - (\partial\mathscr{L}/\partial A^{ij}{}_{\rho,\mu})A^{ij}{}_{\rho,\nu} \tag{3.12}$$

There are two gauge fields $F_{ij}{}^{\mu\rho}$ and $G^{\alpha\beta}{}_\nu$, whose field equations are (3.3) and (3.6). Noether's current $S^\mu{}_{ij}$ is the conserved quantity associated to the invariance of the Lagrangian density under rotations of the vierbein, i.e., the intrinsic spin. Analogously $T^\mu{}_\nu$ is the conserved current originated on the displacements of the vierbein origin; it thus corresponds to the energy tensor. The current $S^\mu{}_{ij}$ creates the field $F_{ij}{}^{\mu\nu}$, whereas the canonical energy tensor $T^\mu{}_\nu$ is only part of the source of the field $G^{\alpha\beta}{}_\nu$; the remaining term, $\partial_\mu(F_{ij}{}^{\rho\mu}A^{ij}{}_\nu)$, has the form of the divergence of an antisymmetric quantity. Therefore, the expression

$$T^{*\rho}{}_\nu \equiv T^\rho{}_\nu - \partial_\mu(F_{ij}{}^{\rho\mu}A^{ij}{}_\nu) \tag{3.13}$$

is the canonical energy tensor when a total divergence is added to the Lagrangian. It satisfies the equation

$$\partial_\rho T^{*\rho}{}_\nu = \partial_\rho T^\rho{}_\nu = 0 \tag{3.14}$$

because the divergence of the second term vanishes identically. This result is consistent with equations (3.4) and (3.7) that imply

$$G^{\alpha\beta}{}_\nu = -G^{\beta\alpha}{}_\nu \tag{3.15}$$

whence, by virtue of equation (3.6), equation (3.14) obtains.

In the next section, using a quadratic Lagrangian density, we identify the fields $F_{ij}{}^{\mu\rho}$ and $G^{\alpha\beta}{}_\nu$ with the curvature and torsion of space-time.

4. *Free Gravitational Lagrangian*

From the general theory of gauge fields (Utiyama, 1956), we know that the free Lagrangian density associated to the extended Poincaré invariance (2.5) contains the derivatives $A^{ij}{}_{\mu,\nu}$ and $h_k{}^\mu{}_{,\nu}$ only through the covariant combinations $R^i{}_{jkl}$ and $C^i{}_{kl}$. The close analogy between formulas (2.11) and (2.16) suggests us that this free Lagrangian should be a quadratic expression in both tensors. This is not the simplest gauge invariant Lagrangian, however. In fact, with the curvature scalar R we can build the following Lagrangian density, linear in the derivatives $A^{ij}{}_{\mu,\nu}$:

$$\mathscr{L}_g = (\det h_k{}^\mu)^{-1} R^{ij}{}_{ij} \tag{4.1}$$

When this expression is written in covariant language and use is made of equations (2.17) and (2.18) it reduces to Palatini's Lagrangian (1.2). The corresponding equations of motion are given by

$$R_{ij} \equiv R_i{}^k{}_{jk} = 0 \tag{4.2}$$

$$C^k{}_{ij} = 0 \tag{4.3}$$

The first is Einstein's gravity equation and the second one expresses the absence of torsion in empty space. Note that both equations are *algebraic* in the gauge fields. The fact that we do not obtain *differential* equations in the curvature and torsion tensors is a serious objection against Lagrangian (4.1) in this gauge approach. As pointed out in the Introduction, Palatini's Lagrangian is essentially a quadratic form when the gravitational field is identified with Christoffel's connexion $\Gamma^\lambda{}_{\mu\nu}$.

In the light of the previous discussion, we propose to adopt as the free gravitational Lagrangian, the quadratic scalar density

$$\mathscr{L}_g = \mathscr{L}_R + \mathscr{L}_C = (\tfrac{1}{4})(\det h_k{}^\mu)^{-1} \times [R^i{}_{jkl} R_i{}^{jkl} + C^i{}_{jk} C_i{}^{jk}] \tag{4.4}$$

where $\mathscr{L}_R$ and $\mathscr{L}_C$ stand for the terms quadratic in Riemann's and Cartan's tensors, respectively. From this Lagrangian we can derive explicit expressions for the gauge fields introduced in Section 3. Writing h for det $h_k{}^\mu$, we obtain

$$(F_g)_{ij}{}^{\mu\nu} \equiv (\partial \mathscr{L}_g / \partial A^{ij}{}_{\mu,\nu}) = h^{-1} R_{ij}{}^{\mu\nu} \tag{4.5}$$

$$(G_g)^{\alpha\beta}{}_\nu \equiv (\partial \mathscr{L}_g/\partial h_k{}^\nu{}_{,\alpha})h_k{}^\beta = h^{-1}C_\nu{}^{\alpha\beta} \tag{4.6}$$

Therefore, the gauge fields F_g and G_g (associated to the potentials $A^{ij}{}_\mu$ and $h_k{}^\nu$) are identified, up to the factor h^{-1}, with the curvature $R_{ij}{}^{\mu\nu}$ and the torsion $C_\nu{}^{\alpha\beta}$.

Note that for Palatini's Lagrangian (4.1), the corresponding gauge fields are

$$(F_g)_{ij}{}^{\mu\nu} = h^{-1}(h_i{}^\mu h_j{}^\nu - h_i{}^\nu h_j{}^\mu)$$
$$(G_g)^{\alpha\beta}{}_\nu = 0$$

Inserting now equations (4.5) and (4.6) into equations (3.10) and (3.13) and making use of the equations of motion derived from equations (4.4), we determine the values of the intrinsic spin and energy densities, which are given by

$$(S_g)^\mu{}_{ij} \equiv (\Sigma_C)^\mu{}_{ij} + (\sigma_R)^\mu{}_{ij} \tag{4.7}$$

$$(T_g{}^*)^\rho{}_\nu \equiv (T_R)^\rho{}_\nu + (T_C)^\rho{}_\nu + (t_C)^\rho{}_\nu \tag{4.8}$$

with

$$(\Sigma_C)^\mu{}_{ij} \equiv \partial \mathscr{L}_C/\partial A_{ij\mu} = \tfrac{1}{2}h^{-1}[C_{ij}{}^\mu - C_{ji}{}^\mu] \tag{4.9}$$

$$(\sigma_R)^\mu{}_{ij} \equiv \partial \mathscr{L}_R/\partial A_{ij\mu} = h^{-1}(A_{ia\rho}R_j{}^{a\mu\rho} - A_{ja\rho}R_i{}^{a\mu\rho}) \tag{4.10}$$

$$(T_R)^\rho{}_\nu \equiv (\partial \mathscr{L}_R/\partial h_k{}^\nu)h_k{}^\rho = h^{-1}(R_{ij}{}^{\mu\rho}R^{ij}{}_{\mu\nu} - \tfrac{1}{4}\delta^\rho{}_\nu R^i{}_{jkl}R_i{}^{jkl}) \tag{4.11}$$

$$(T_C)^\rho{}_\nu + (t_C)^\rho{}_\nu \equiv (\partial \mathscr{L}_C/\partial h_k{}^\nu)h_k{}^\rho + (\partial \mathscr{L}_C/\partial h_k{}^\nu{}_{,\mu})h_k{}^\rho{}_{,\mu} \tag{4.12}$$

$$(T_C)^\rho{}_\nu = h^{-1}(C_{ij}{}^\rho C^{ij}{}_\nu - \tfrac{1}{4}\delta^\rho{}_\nu C^i{}_{jk}C_i{}^{jk}) \tag{4.13}$$

$$(t_C)^\rho{}_\nu = C_\mu{}^{\epsilon\rho}(b^k{}_\nu h_k{}^\mu{}_{,\epsilon} - A^\mu{}_{\nu\epsilon}) \tag{4.14}$$

The intrinsic spin of the gravitational field (4.7) naturally decomposes into a tensor part (4.9), given by the spin of the torsion field, and a pseudotensor (4.10), which corresponds to the spin of the curvature field. The intrinsic energy (4.8) splits into three terms: a tensor part (4.11), given by the energy of the curvature field; another tensor (4.13), and a pseudotensor (4.14), which correspond to the energy of the torsion field.

We are now in a position to write the equations of motion. From the Lagrangian density (4.4) and taking into account equations (3.3) and (3.6) we obtain

$$\partial_\nu(h^{-1}R_{ij}{}^{\mu\nu}) + h^{-1}A_{ja\nu}R_i{}^{a\mu\nu} - h^{-1}A_{ia\nu}R_j{}^{a\mu\nu} = (\Sigma_C)^\mu{}_{ij} \tag{4.15}$$

$$\partial_\nu(h^{-1}C_\alpha{}^{\mu\nu}) - C_\nu{}^{\epsilon\mu}(b^k{}_\alpha h_k{}^\nu{}_{,\epsilon} - A^\nu{}_{\alpha\epsilon}) = (T_R)^\mu{}_\alpha + (T_C)^\mu{}_\alpha \tag{4.16}$$

We have relegated the pseudotensors to the left-hand side of these equations, because their role consists in changing the ordinary divergence into a covariant one.

5. *Covariant Equations and Gravitational Energy*

The equations of motion (4.15) and (4.16) for the curvature and torsion fields may be cast in covariant language with the help of equations (2.17) and (2.18). The determinant h is easily seen to be equal to $(-g)^{-1/2}$ with $g = \det(g_{\mu\nu})$. A straightforward calculation then yields

$$R^{\alpha\beta\mu\nu}{}_{;\nu} - \tfrac{1}{2}C^{\mu}{}_{\epsilon\nu}R^{\alpha\beta\epsilon\nu} - \tfrac{1}{2}C^{\nu}{}_{\epsilon\nu}R^{\alpha\beta\mu\epsilon} = (\Sigma_C)^{\mu\alpha\beta} \tag{5.1}$$

$$C^{\beta\mu\nu}{}_{;\nu} - \tfrac{1}{2}C^{\mu}{}_{\epsilon\nu}C^{\beta\epsilon\nu} - \tfrac{1}{2}C^{\nu}{}_{\epsilon\nu}C^{\beta\mu\epsilon} = (T_R)^{\beta\mu} + (T_C)^{\beta\mu} \tag{5.2}$$

On the left-hand side of these equations, the presence of torsion introduces two additional terms, which are characteristic of the divergence of an antisymmetric tensor density calculated with a nonsymmetric connexion. In fact, if $\mathscr{A}^{\mu\nu} = -\mathscr{A}^{\nu\mu}$ is an antisymmetric tensor density, i.e., if it transforms like $(-g)^{1/2}A^{\mu\nu}$, where $A^{\mu\nu}$ is an antisymmetric tensor, then its divergence is given by

$$\partial_\nu \mathscr{A}^{\mu\nu} = \mathscr{A}^{\mu\nu}{}_{;\nu} - \tfrac{1}{2}C^{\mu}{}_{\epsilon\nu}\mathscr{A}^{\epsilon\nu} - \tfrac{1}{2}C^{\nu}{}_{\epsilon\nu}\mathscr{A}^{\mu\epsilon} \tag{5.3}$$

This formula generalizes for non-Riemannian spaces the well-known tensor structure of the ordinary divergence of an antisymmetric second-order tensor density.

The new feature we note here at once is the presence of torsion in empty space. Its source is the sum of the energy tensors of the curvature and torsion fields. In covariant notation they become

$$(T_R)^{\beta\mu} = R^{\xi}{}_{\eta\gamma}{}^{\beta}R_{\xi}{}^{\eta\gamma\mu} - \tfrac{1}{4}g^{\beta\mu}R^{\xi}{}_{\eta\gamma\delta}R_{\xi}{}^{\eta\gamma\delta} \tag{5.4}$$

$$(T_C)^{\beta\mu} = C^{\xi}{}_{\eta}{}^{\beta}C_{\xi}{}^{\eta\mu} - \tfrac{1}{4}g^{\beta\mu}C^{\xi}{}_{\eta\gamma}C_{\xi}{}^{\eta\gamma} \tag{5.5}$$

Note the similarity between these expressions with Maxwell's energy tensor of electrodynamics. They are quadratic in the fields, symmetric, and traceless. On the other hand, the source of the curvature field is the torsion spin, which is given by

$$(\Sigma_C)^{\mu\alpha\beta} \equiv \tfrac{1}{2}(C^{\alpha\beta\mu} - C^{\beta\alpha\mu}) \tag{5.6}$$

In the particular case when no torsion is present, the field equations (5.1) and (5.2) reduce to

$$R^{\alpha\beta\mu\nu}{}_{;\nu} = 0 \tag{5.7}$$

$$(T_R)^{\beta\mu} = 0 \tag{5.8}$$

Equation (5.7) is equivalent to the once contracted Bianchi identities, combined with Einstein's equation $R^{\mu\nu} = 0$ (López, 1973), and equation (5.8) is a well-known property of the curvature tensor in an empty ($R^{\mu\nu} = 0$) Riemannian space-time (Lanczos, 1938). Therefore, in Riemannian spaces the dynamical gravitational energy vanishes. This implies that a gravitational wave is characterized by besides the curvature, the torsion of space-time.

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